Brownian motion with killing and reflection and the "hot–spots" problem

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Abstract

We investigate the "hot–spots" property for the survival time probability of Brownian motion with killing and reflection in planar convex domains whose boundary consists of two curves, one of which is an arc of a circle, intersecting at acute angles. This leads to the "hot–spots" property for the mixed Dirichlet–Neumann eigenvalue problem in the domain with Neumann conditions on one of the curves and Dirichlet conditions on the other.

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1 Introduction

The "hot spots" conjecture, formulated by J. Rauch in 1974, asserts that the maximum and the minimum of the first nonconstant Neumann eigenfunction for a smooth bounded domain in $\mathbb{R}^n$ are attained on the boundary and only on the boundary (see [4] for more precise formulation). The conjecture has received a lot of attention in recent years and partial results have been obtained in [12], [4], [11], [1], [2], [13]. Counterexamples for (nonconvex) domains in the plane and on surfaces have been given in [8], [7] and [10]. We refer the reader to [5] where a different proof of the result in [13] is given and for more details on the above literature.

The conjecture is widely believed to be true for arbitrary convex domains in the plane but surprisingly even this remains open. For planar convex domains (and indeed for any simply connected domain) the conjecture can be formulated in terms of a mixed Dirichlet–Neumann eigenvalue problem as discussed in [5]. The purpose of this note is to explore this mixed boundary value problem further and in particular to extend the results in [13] and [5].

We assume for the rest of the paper that D is a planar convex domain for which the Laplacian with Neumann boundary conditions has discrete spectrum. The eigenvalues of the Laplacian are a sequence of nonnegative numbers tending to infinity and 0 is always an eigenvalue with eigenfunction 1. Let μ_1 be the first nonzero eigenvalue. Under various conditions on D, it is shown in [7] that μ_1 is simple. In general the multiplicity of μ_1 is at most 2 (see [7]). Let φ_1 be any Neumann eigenfunction corresponding to μ_1. The strongest form of the "hot–spots" conjecture (see [4] for other weaker forms) asserts that φ_1 attains its maximum on $\overline{D}$ on, and only, ∂D.

The set $\gamma = \overline{\{x \in D : \varphi_1(x) = 0\}}$ is called the *nodal line for* φ_1. It follows from Pólya's comparisons of Dirichlet and Neumann eigenvalues that φ_1 does not have closed nodal lines. That is, γ a smooth simple curve intersecting the boundary at exactly two points and divides the domain into two simply connected domains D_1 and D_2, called *nodal domains*. We can take $\varphi_1 > 0$ on D_1 and $\varphi_1 < 0$ on D_2. The function φ_1 is an eigenfunction corresponding to the smallest eigenvalue for the Laplacian in D_1 with Dirichlet boundary conditions on γ and Neumann boundary conditions on $\partial D_1 \backslash \gamma$. The "hot–spots" conjecture is equivalent to the assertion that this function takes its maximum on, and only on, $\partial D_1 \backslash \gamma$.

The results in [13] and [5] can be stated in terms of the above mixed Dirichlet-Neumann boundary value problem as follows. Suppose that D is planar convex domain whose boundary consists of the curve γ_1 and the line segment γ_2. Let μ_1 be the lowest eigenvalue for the Laplacian in D with

Neumann boundary conditions on γ_1 and Dirichlet boundary conditions on γ_2. Let $\psi_1 : \overline{D} \to [0, \infty)$ be the ground state eigenfunction (unique up to a multiplicative constant) corresponding to μ_1. Then ψ_1 attains its maximum on, and only on, γ_1. In fact, the results in [13], [5] prove more. Let B_t be a reflecting Brownian motion in D starting at $z \in \overline{D}$ which is killed on γ_2, and let τ denote its lifetime (the first time B_t hits γ_2). Then, for an arbitrarily fixed $t > 0$, the function $u(z) = P^z\{\tau > t\}$ attains is maximum, as a function of $z \in \overline{D}$, on, and only on, γ_1. Furthermore, both function $u(z)$ and $\psi_1(z)$ are strictly increasing as z moves toward the boundary γ_1 of D along hyperbolic line segments. (See [13] and [5] for the precise definitions of hyperbolic line segments and for the details of how the result for u implies the result for ψ_1.) The following question, first raised in [5], naturally arises:

Question. *Given a bounded simply connected planar domain whose boundary consists of two smooth curves, what conditions must one impose on these two curves in order for the ground state eigenfunction of the mixed boundary value problem (Dirichlet conditions on one curve and Neumann on the other) to attain its maximum on the boundary and only on the boundary?*

In this paper we prove the following theorem which extends the results in [13] and [5] by replacing the hypothesis that γ_2 is a line segment by the hypothesis that γ_2 is an arc of a circle.

Theorem 1.1. *Suppose D is a bounded convex planar domain whose boundary consists of two curves $\{\gamma_1(t)\}_{t\in[0,1]}$ and $\{\gamma_2(t)\}_{t\in[0,1]}$ one of which is an arc of a circle, and suppose that the angle between the curves γ_1 and γ_2 is less than or equal to $\frac{\pi}{2}$. That is, the angle formed by the two half-tangents at $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ is less than or equal to $\frac{\pi}{2}$. Let B_t be a reflecting Brownian motion in D killed on γ_2 and let τ_D denote its lifetime. Then, for each $t > 0$ arbitrarily fixed, the function $u(z) = P^z\{\tau_D > t\}$ attains it maximum on, and only on, γ_1.*

Corollary 1.2. *("Hot–spots" for the mixed boundary value problem.) Let D be as in Theorem 1.1. Let ψ_1 be a first mixed Dirichlet-Neumann eigenfunction for the Laplacian in D, with Neumann boundary conditions on γ_1 and Dirichlet boundary conditions on γ_2. Then $\psi_1(z)$, $z \in \overline{D}$, attains its maximum on, and only on, γ_1.*

As in [13] and [5], the functions $u(z)$ and $\psi_1(z)$ are increasing along hyperbolic line segments in D, in the case when γ_2 is an arc of a circle and along Euclidean radii contained in D in the case when γ_1 is an arc of a circle. We shall make this precise later. The proof of Theorem 1.1

is presented in the next section. The idea for the case when γ_2 is an arc of a circle is to construct a convex domain starting from D, by symmetry with respect to a circle (the circle which contains the arc γ_2), and then use the stochastic inequality for potentials proved in [5]. This inequality also follows from the coupling arguments in [13] which have the advantage that they work in several dimensions. Hence we will discuss this inequality in several dimensions. While at this point we have no applications for this more general inequality, we believe the inequality is of independent interest. The case when γ_1 is an arc of a circle is treated by a coupling argument right in the domain itself.

2 Preliminary Results

The proof of Theorem 1.1 is different depending on which one of curves γ_1 or γ_2 is an arc of a circle. For the proof of the case when γ_2 is an arc of a circle, we need several preliminary results.

Proposition 2.1. *Let D be as in Theorem 1.1 and suppose that γ_2 is an arc of a circle $C = \partial B(z_0, R)$. Let D_s be the domain which is symmetric to the domain D with respect to the circle C, that is*

$$D_s = \{z_0 + \frac{R^2}{\overline{z} - \overline{z_0}} : z \in D\}.$$

Then $D^* = D \cup \gamma_2 \cup D_s$ is a convex domain.

Proof. For a complex number z we will use $\Re z$ and $\Im z$ to denote the real, respectively the imaginary part of the complex number $z \in \mathbb{C}$. Without loss of generality we can assume that $C = \partial B(0, 1)$ is the circle centered at the origin of radius 1 and that $\gamma_1(0)$ and $\gamma_1(1)$ are symmetric with respect to the vertical axis, that is $\Im\gamma_1(0) = \Im\gamma_2(1)$. Further, we may assume that γ_2 contains the point $-i$.

We will first show that $\Im\gamma_1(0) \leq 0$. To see this, note that since the domain D is convex, it lies below its half-tangent at the point $\gamma_1(0)$, and by the angle restriction this half-line lies below the line passing through $\gamma_1(0)$ and the origin. If $\Im\gamma_1(0) > 0$ then also $\Im\gamma_1(1) = \Im\gamma_1(0) > 0$, and therefore the point $\gamma_1(1) \in \partial D$ does not lie below (or on) the line determined by $\gamma_1(0)$ and 0, a contradiction. We must therefore have $\Im\gamma_1(0) = \Im\gamma_1(1) \leq 0$.

If $\Im\gamma_1(0) = \Im\gamma_1(1) = 0$, by the angle restriction at these points, together with the fact that D is a convex domain (and hence γ_1 is a concave down curve), it follows that the curve γ_1 is in this case the line segment $[-1, 1]$,

and therefore $D = \{z \in \mathbb{C} : \Im z < 0, |z| < 1\}$. The proof is trivial in this case since $D_s = \{z \in \mathbb{C} : \Im z < 0, |z| > 1\}$, and therefore $D^* = D \cup \gamma_2 \cup D_s = \{z \in \mathbb{C} : \Im z < 0\}$ which is a convex domain.

A similar argument shows that if $0 \in \gamma_1 \subset \partial D$, then the curve γ_1 consists of the union of the two line segments from $\gamma_1(0)$ to 0, respectively from 0 to $\gamma_1(1)$, hence D is a sector of the unit disk. It follows that $D^* = D \cup \gamma_2 \cup D_s = \{z \in \mathbb{C} - \{0\} : \arg \gamma_1(0) < \arg z < \arg \gamma_1(1)\}$, which is again a convex set. We can therefore assume that $\Im \gamma_1(0) = \Im \gamma_1(1) < 0$ and $0 \notin D \cup \partial D$. It follows that the domain D is contained in the circular sector $\{z \in \mathbb{C} - \{0\} : |z| < 1, \arg \gamma_1(0) < \arg z < \arg \gamma_1(1)\}$, and therefore $D^* = D \cup \gamma_2 \cup D_s$ is contained in $\{z \in \mathbb{C} - \{0\} : \arg \gamma_1(0) < \arg z < \arg \gamma_1(1)\}$. It follows that for any points $w_1, w_2 \in D^* = D \cup \gamma_2 \cup D_s$, the line segment $[w_1, w_2]$ may intersect the circle C only on the arc γ_2 (and not on $C - \gamma_2$). Since D is convex domain, it follows that $D^* = D \cup \gamma_2 \cup D_s$ is a convex domain if and only if

$$(2.1)\quad w_1 \in D_s, w_2 \in \gamma_2 \cup D_s \text{ s.t. } [w_1, w_2] \cap \gamma_2 \in \{\emptyset, \{w_2\}\} \Rightarrow [w_1, w_2] \subset D^*,$$

where $[w_1, w_2]$ denotes the line segment with endpoints w_1 and w_2.

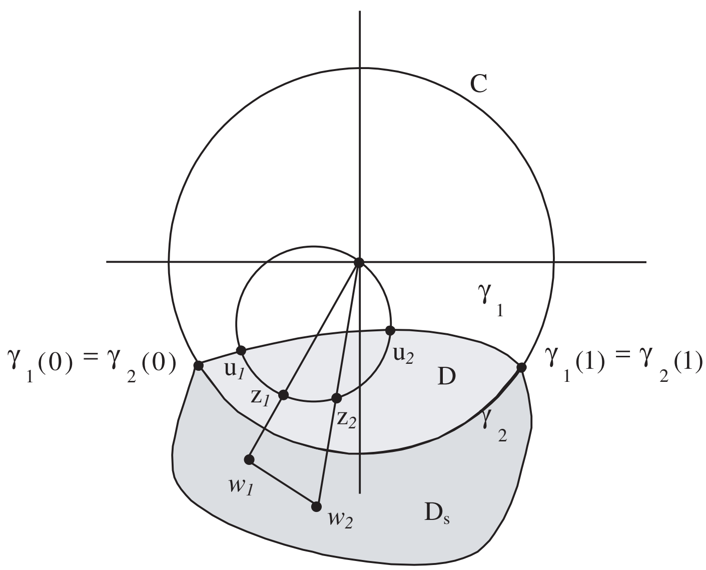

Figure 1

Since the set is symmetric to a line with respect to C is a circle passing through the origin, by letting z_1, z_2 be the symmetric points of w_1, respec-

tively w_2 with respect to C, (2.1) can be rewritten equivalently as

$$\text{(2.2)} \qquad z_1 \in D, z_2 \in \gamma_2 \cup D \text{ s.t. } \widehat{z_1 z_2} \cap \gamma_2 \in \{\emptyset, \{z_2\}\} \Rightarrow \widehat{z_1 z_2} \subset \gamma_2 \cup D,$$

where $\widehat{z_1 z_2}$ denotes the arc of the circle $C(0, z_1, z_2)$ passing through z_1, z_2 and 0, between (and including) z_1 and z_2, and not containing 0. If the points z_1, z_2 and 0 are collinear, the arc $\widehat{z_1 z_2}$ becomes the line segment $[z_1, z_2]$.

To show the claim, we will prove (2.2). Let $z_1 \in D$, $z_2 \in \gamma_2 \cup D$ such that $\widehat{z_1 z_2} \cap \gamma_2 \in \{\emptyset, \{z_2\}\}$. If the points $0, z_1$ and z_2 are collinear, $\widehat{z_1 z_2} = [z_1, z_2] \subset \gamma_2 \cup D$, so we may assume that $0, z_1$ and z_2 are not collinear.

Assume first that the circle $C(0, z_1, z_2)$ does not intersect C. Since γ_1 bounds the convex domain D, the intersection $\gamma_1 \cap C(0, z_1, z_2)$ consists of exactly two points u_1 and u_2 (see Figure 1). It follows that the intersection between D and $C(0, z_1, z_2)$ is the arc $\widehat{u_1 u_2}$, and therefore we have $\widehat{z_1 z_2} \subset \widehat{u_1 u_2} \subset D$ in this case.

If the circle $C(0, z_1, z_2)$ intersects C, the intersection $C(0, z_1, z_2) \cap D$ is either one or two (connected) arcs c_1 and c_2. Note that z_1 and z_2 must lie on the same connected arc c_i ($i = 1$ or $i = 2$), for otherwise the intersection $\widehat{z_1 z_2} \cap \gamma_2$ would consist of two distinct points (the two endpoints of c_1 and c_2 lying on γ_2). If $z_1, z_2 \in c_1$, since c_1 is a connected arc lying in D, we have $\widehat{z_1 z_2} \subset c_1 \cup \gamma_2 \subset D \cup \gamma_2$ and the claim follows. This completes the proof of the Proposition. □

Using the Schwarz reflection principle and the above lemma, we can prove the following

Corollary 2.1. *Let D be as in Theorem 1.1 and suppose that γ_2 is an arc of a circle. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $U^+ = \{z \in U : \Im z > 0\}$ be the upper half-disk. Let $f : \overline{U^+} \to \overline{D}$ be a conformal map such that $f[-1, 1] = \gamma_2$. Then f extends to a conformal map from U onto the convex domain D^*.*

Proof. Assume γ_2 is an arc of a circle $\partial B(z_0, r)$ of radius r centered at z_0.

Consider the function $\widetilde{f} : U \to \mathbb{C}$ defined by

$$\widetilde{f}(z) = \begin{cases} f(z), & z \in \overline{U^+} \\ \\ z_0 + \frac{r^2}{\overline{f(\overline{z})} - z_0}, & z \in U \backslash \overline{U^+} \end{cases}.$$

Since f maps the line segment $[-1, 1]$ onto the arc γ_2 of the circle $\partial B(z_0, r)$, by the Schwarz symmetry principle it follows that $\widetilde{f}$ is a conformal extension of f, from the unit disk U onto the domain $D^* = D \cup \gamma_2 \cup D_s$, which by Proposition 2.1 is a convex domain. □

Corollary 2.2. *If f is as in Corollary 2.1, then for any $\theta \in [0, 2\pi)$ arbitrarily fixed, $r\left|f'(re^{i\theta})\right|$ is an increasing function of $r \in (0,1)$.*

Proof. As in [13], we have:

$$\begin{aligned} \frac{\partial}{\partial r} \ln r\left|f'(re^{i\theta})\right| &= \frac{1}{r} + \frac{\partial}{\partial r}\Re(\ln f'(re^{i\theta})) \\ &= \frac{1}{r} + \Re\frac{\partial}{\partial r}\ln f'(re^{i\theta}) \\ &= \frac{1}{r} + \Re(e^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}) \\ &= \frac{1}{r}\Re(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}), \end{aligned} \tag{2.3}$$

for any $r \in (0,1)$ and $\theta \in (0, 2\pi)$.

By the above proposition, f extends to a convex map $f : U \to D^*$; it is known (see [9]) that any convex map $f : U \to \mathbb{C}$ satisfies the inequality

$$\Re(1 + z\frac{f''(z)}{f'(z)}) > 0, \quad z \in U,$$

which shows that the quantity on the right side of (2.3) is strictly positive, and therefore $\ln r\left|f'(re^{i\theta}\right|$ is a strictly increasing function of $r \in (0,1)$ for any $\theta \in [0, 2\pi)$ arbitrarily fixed, which proves the claim. □

To complete the proof of Theorem 1.1 in the case when γ_2 is an arc of a circle we will use the following theorem which may be of independent interest.

Theorem 2.3. *Let $U_d = \{\zeta \in \mathbb{R}^d : \|\zeta\| < 1\}$ be the unit ball in $\mathbb{R}^d$, $d \geq 2$, and let $U_d^+ = \{\zeta = (\zeta_1, ...\zeta_d) \in U_d : \zeta_n > 0\}$ be the upper hemisphere in $\mathbb{R}^d$. Suppose that $V : \overline{U_d^+} \to (0, \infty)$ is a continuous potential for which $r^2V(r\zeta)$ is a nondecreasing function of $r \in (0, \frac{1}{\|\zeta\|})$ for any $\zeta \in U_d^+$ arbitrarily fixed. That is, suppose that*

$$r_1^2 V(r_1\zeta) \leq r_2^2 V(r_2\zeta), \tag{2.4}$$

for all $\zeta \in U_d^+$, $0 < r_1 < r_2 < \frac{1}{\|\zeta\|}$. Let B_t be a reflecting Brownian motion in U_d^+ killed on the hyperplane $H = \{\zeta = (\zeta_1, ..., \zeta_d) \in \mathbb{R}^d : \zeta_d = 0\}$, and let $\tau_{U_d^+}$ denote its lifetime. Then for any arbitrarily fixed $t > 0$ and $\zeta \in U_d^+$,

$P^{r\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t\right\}$ *is a nondecreasing function of* $r \in (0, \frac{1}{\|\zeta\|})$. *That is,*

$$P^{r_1\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t\right\} \leq P^{r_2\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t\right\}, \tag{2.5}$$

for all $t > 0$, $\zeta \in U_d^+$ *and*

$$0 < r_1 < r_2 < \frac{1}{\|\zeta\|}.$$

Moreover, if the inequality in (2.4) is a strict inequality, so is the one in (2.5).

Remark 2.1. *For* $d = 2$, *the Proposition as stated is proved in [5]. It also follows from the arguments in [13]. However, the proof in [13] can be made to work for all* $d \geq 2$ *and this is the argument we follow here.*

Proof. Fix $t > 0$, $\zeta \in U_d^+$ and $0 < r_1 < r_2 < \frac{1}{\|\zeta\|}$. Following [13], we consider a scaling coupling of reflecting Brownian motions $(B_t, \widetilde{B}_t)$ in the unit ball U_d starting at $(r_1\zeta, r_2\zeta)$. More precisely, let B_t be reflecting Brownian motion in U_d starting at $r_1\zeta \in U_d$, with its natural filtration $\mathcal{F}_t$, and consider

$$\widetilde{B}_t = \frac{1}{M_{\alpha_t}} B_{\alpha_t}, \quad t \geq 0, \tag{2.6}$$

where

$$M_t = \frac{r_1}{r_2} \vee \sup_{s \leq t} \|B_s\|, \tag{2.7}$$

$$A_t = \int_0^t \frac{1}{M_s^2} ds, \tag{2.8}$$

and

$$\alpha_t = \inf\{s > 0 : A_s \geq t\}. \tag{2.9}$$

Theorem 2.3 and Remark 2.4 of [13] show that $\widetilde{B}_t$ is an $(\mathcal{F}_{\alpha_t})$-adapted reflecting Brownian in U_n.

Letting $\tau_{U_d^+}$, $\widetilde{\tau}_{U_d^+}$ denote the killing times of B_t, respectively $\widetilde{B}_t$, on the hyperplane $H = \{\zeta = (\zeta_1, ..., \zeta_d) \in \mathbb{R}^d : \zeta_d = 0\}$, we have almost surely

$$\begin{aligned}
\tau_{U_d^+} &= \inf\{s > 0 : B_s \in H\} \\
&= \inf\{\alpha_u > 0 : B_{\alpha_u} \in H\} \\
&= \inf\{\alpha_u > 0 : \widetilde{B}_u \in H\} \\
&= \alpha_{\inf\{u>0:\widetilde{B}_u \in H\}} \\
&= \alpha_{\widetilde{\tau}_{U_d^+}},
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}
\int_0^{\tau_{U_d^+}} V(B_s)ds &= \int_0^{\alpha_{\widetilde{\tau}_{U_d^+}}} V(B_s)ds \\
&= \int_0^{\widetilde{\tau}_{U_d^+}} V(B_{\alpha_u})d\alpha_u \\
&= \int_0^{\widetilde{\tau}_{U_d^+}} V(B_{\alpha_u})M_{\alpha_u}^2 du \\
&\leq \int_0^{\widetilde{\tau}_{U_d^+}} V(\frac{1}{M_{\alpha_u}} B_{\alpha_u})du \\
&= \int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_u)du.
\end{aligned} \tag{2.10}$$

The inequality above follows from the assumption that $r^2V(r\zeta)$ is a nondecreasing function of r for $\zeta \in U_d^+$ arbitrarily fixed:

$$V(B_{\alpha_u}) = 1^2 V(1B_{\alpha_u}) \leq \frac{1}{M_{\alpha_u}^2} V(\frac{1}{M_{\alpha_u}} B_{\alpha_u}),$$

since by (2.7) we have $M_{\alpha_u} \leq 1$ for all $u \geq 0$.

By the construction above, $(B_t, \widetilde{B}_t)$ is a pair of reflecting Brownian motions in U_d starting at $(r_1\zeta, r_2\zeta)$, and the inequality (2.10) shows that we have in particular

$$P^{r_1\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t\right\} \leq P^{r_2\zeta}\left\{\int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds > t\right\},$$

which proves the first part of the Theorem (2.3).

To prove the strict increasing part of the theorem, we will use the following support lemma for the n-dimensional Brownian motion (see [14], page 374.)

Lemma 2.4. *Given an d-dimensional Brownian motion B_t starting at x and a continuous function $f : [0, 1] \to \mathbb{R}^d$ with $f(0) = x$ and $\varepsilon > 0$, we have*

$$P^x(\sup_{t\leq 1} \|B_t - f(t)\| < \varepsilon) > 0.$$

Assume now that we have strict inequality in (2.4). By the continuity of the potential $V : \overline{U_d^+} \to (0, \infty)$ and the strict monotonicity of $r^2 V(r\zeta)$ for $0 < r < \frac{1}{\|\zeta\|}$, we have

$$\int_0^1 V((1-u)r_1\zeta)du < \int_0^1 \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1}(1-u)r_1\zeta)du,$$

and therefore we can choose $T > 0$ such that

$$T\int_0^1 V((1-u)r_1\zeta)du < t < T\int_0^1 \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1}(1-u)r_1\zeta)du,$$

and we may further choose $\varepsilon > 0$ and $\delta > 0$ small enough so that

$$\frac{T}{1+\delta}\int_0^{1+\frac{\varepsilon}{r_1}} V((1-u)r_1\zeta)du < t < \frac{T}{1+\delta}\int_0^{1-\frac{\varepsilon}{r_1}} \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1}(1-u)r_1\zeta)du. \tag{2.11}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}^d$ defined by

$$f(s) = (1 - \frac{(1+\delta)}{T}s)r_1\zeta.$$

With the change of variable $u = \frac{1+\delta}{T}s$, the double inequality in (2.11) can be rewritten as

$$\int_0^{\frac{1+\frac{\varepsilon}{r_1}}{1+\delta}T} V(f(s))ds < t < \int_0^{\frac{1-\frac{\varepsilon}{r_1}}{1+\delta}T} \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1}f(s))ds.$$

By eventually choosing a smaller $\varepsilon > 0$, and by the uniform continuity of V on $\overline{U^+}$, we also have

$$\int_0^{\frac{1+\frac{\varepsilon}{r_1}}{1+\delta}T} V(b(s))ds < t < \int_0^{\frac{1-\frac{\varepsilon}{r_1}}{1+\delta}T} \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1}b(s))ds, \tag{2.12}$$

for any continuous functions $b : [0, \frac{T}{1+\delta}] \to \mathbb{R}^n$ such that

$$\sup_{s\leq\frac{T}{1+\delta}} \|b(s) - f(s)\| < \varepsilon.$$

Let B_t and $\tilde{B}_t$ be the reflecting Brownian motions in U_d starting at $r_1\zeta$, respectively $r_2\zeta$, as constructed above. By Lemma (2.4), B_t lies in the ε-tube about $f(t)$ for $0 < t < T$ with positive probability. That is,

$$P(\sup_{s\leq T}|B_s - f(s)| < \varepsilon) > 0.$$

We may assume that $\varepsilon > 0$ is chosen small enough so that this tube does not intersect ∂U, and therefore on a set Q of positive probability, the coupled Brownian motion $\widetilde{B}_s$ does not reach ∂U_d, hence the process M_s is constant on this set. Thus, on Q we have

$$M_s = \frac{r_1}{r_2}, \tag{2.13}$$

$$A_s = \int_0^s \frac{1}{M_u^2} du = \left(\frac{r_2}{r_1}\right)^2 s \tag{2.14}$$

$$\alpha_s = A_s^{-1} = \left(\frac{r_1}{r_2}\right)^2 s. \tag{2.15}$$

and $\widetilde{\tau}_{U_d^+} = A_{\tau_{U_d^+}} = \left(\frac{r_2}{r_1}\right)^2 \tau_{U_d^+}$. Therefore on Q we have

$$\begin{aligned} \int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds &= \int_0^{\left(\frac{r_2}{r_1}\right)^2 \tau_{U_d^+}} V(\frac{1}{M_{\alpha_s}} B_{\alpha_s})ds \\ &= \int_0^{\tau_{U_d^+}} \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1} B_u)du \\ &> \int_0^{\tau_{U_d^+}} V(B_s)ds. \end{aligned} \tag{2.16}$$

Also, by the construction of the set Q we have $\frac{1-\frac{\varepsilon}{r_1}}{1+\delta}T < \tau_{U_d^+} < \frac{1+\frac{\varepsilon}{r_1}}{1+\delta}T$ on Q, and combining with (2.12) and (2.16), we obtain the strict inequality

$$\begin{aligned} \int_0^{\tau_{U_d^+}} V(B_s)ds &\leq \int_0^{T\frac{1+\frac{\varepsilon}{r_1}}{1+\delta}} V(B_s)ds < t \\ &< \int_0^{T\frac{1-\frac{\varepsilon}{r_1}}{1+\delta}} \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1} B_s)ds \\ &\leq \int_0^{\tau_{U_d^+}} \left(\frac{r_2}{r_1}\right)^2 V(\frac{r_2}{r_1} B_s)ds \\ &= \int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds, \end{aligned} \tag{2.17}$$

almost surely on Q.

Therefore we have:

$$\begin{aligned}
P^{r_1\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t\right\} &= P^{r_1\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t, Q\right\} \\
&+ P^{r_1\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t, Q^c\right\} \\
&= 0 + P^{r_1\zeta}\left\{\int_0^{\tau_{U_d^+}} V(B_s)ds > t, Q^c\right\} \\
&\leq P^{r_2\zeta}\left\{\int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds > t, Q^c\right\} \\
&< P^{r_2\zeta}\left\{Q\right\} + P^{r_2\zeta}\left\{\int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds > t, Q^c\right\} \\
&= P^{r_2\zeta}\left\{\int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds > t, Q\right\} \\
&+ P^{r_2\zeta}\left\{\int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds > t, Q^c\right\} \\
&= P^{r_2\zeta}\left\{\int_0^{\widetilde{\tau}_{U_d^+}} V(\widetilde{B}_s)ds > t\right\},
\end{aligned}$$

which proves the strict inequality in (2.5) in the case when the $r^2V(r\zeta)$ is a strictly increasing function of r, ending the proof of Theorem 2.3. □

3 Proof of Theorem 1.1 and Corollary 1.2

For the proof of Theorem 1.1 we will distinguish the two cases.

Case 1. Suppose γ_2 is an arc of a circle.

Let f a the conformal mapping given by Corollary 2.1, and let B_t be a reflecting Brownian motion in U^+ killed on hitting $[-1,1]$, and denote its lifetime by τ_{U^+}. By Corollary 2.2, the potential $V : U^+ \to \mathbb{R}$ defined by $V(z) = |f'(z)|^2$ satisfies the hypothesis of Theorem 2.3, and therefore we have

$$P^{z_1}\left\{\int_0^{\tau_{U^+}} |f'(B_s)|^2 ds > t\right\} \leq P^{z_2}\left\{\int_0^{\tau_{U^+}} |f'(B_s)|^2 ds > t\right\}, \tag{3.1}$$

for all $t > 0$ and $z_1 = r_1 e^{i\theta},\ z_2 = r_2 e^{i\theta}$ with $0 < r_1 < r_2 < 1$ and $0 < \theta < \pi$. By Lévy's conformal invariance of the Brownian motion, this is exactly the same as

$$P^{f(z_1)}\{\tau_D > t\} \leq P^{f(z_2)}\{\tau_D > t\}, \tag{3.2}$$

where τ_D is as in Theorem 1.1. From this it follows that the function $u(z) = P^z\{\tau_D > t\}$ is nondecreasing as z moves toward γ_1 along the curve $\gamma_\theta = f\{re^{i\theta} : 0 < r < 1\}$, for any $\theta \in (0, \pi)$ arbitrarily fixed. This together with the real analyticity of the function $u(z)$ implies that $u(z)$ is in fact strictly increasing along the family of curves $\{\gamma_\theta : 0 < \theta < \pi\}$, which completes the proof of Theorem 1.1 when γ_2 is an arc of a circle.

Case 2. Suppose γ_1 is an arc of a circle.

Without loss of generality we may assume that γ_1 is an arc of the unit circle centered at the origin. An argument similar to the one in Proposition 2.1 shows that $0 \notin D$, and if $0 \in \partial D$ then the domain D is a sector of the unit disk. It either case, the origin belongs to $U \backslash D$.

We claim that $U \backslash D$ is starlike with respect to the origin. If $0 \in \partial D$, the set D is a sector of the unit disk U and the claim follows. We can assume therefore that $0 \notin \overline{D}$. By the angle restriction in the hypothesis of our theorem, together with the convexity of the domain, it follows that D is contained in a sector of the unit disk U, which without loss of generality may be assumed to be symmetric with respect to the imaginary axis. That is, $D \subset \{z \in U : \alpha < \arg z < \pi - \alpha\}$, where $\alpha = \min\{\arg \gamma_1(0), \arg \gamma_1(1)\} \in (0, \frac{\pi}{2})$. Let $z \in U \backslash D$ and $t \in [0, 1]$ be arbitrarily fixed. If $\arg z \notin (\alpha, \pi - \alpha)$ then $tz \in U \backslash \{z \in U : \alpha < \arg z < \pi - \alpha\} \subset U \backslash D$. Thus $tz \in U \backslash D$ in this case. If $\arg z \in (\alpha, \pi - \alpha)$ and $tz \notin U \backslash D$, then, since $\frac{1}{|z|} z \in \gamma_1 \subset \overline{D}$, we obtain by the convexity of D that the line segment with endpoints tz and $\frac{1}{|z|} z$ is contained in D, and in particular it follows that $z \in D$, a contradiction. In both cases we obtained that $tz \in U \backslash D$, which proves that $U \backslash D$ is starlike with respect to the origin.

We now follow the proof of Theorem 2.3 in the case $d = 2$. For arbitrarily fixed $t > 0$ and $r_1 e^{i\theta}, r_2 e^{i\theta} \in D$ with $r_1 < r_2$, let $(B_t, \widetilde{B}_t)$ be a scaling coupling of reflecting Brownian motions in the unit disk U starting at $(r_1 e^{i\theta}, r_2 e^{i\theta})$, as in the case of Theorem 1.1. We note that that if for $s > 0$ we have $\frac{1}{M_s} B_s \in \gamma_2 \subset U \backslash D$, then by the starlikeness of the set $U \backslash D$ also $B_s \in U \backslash D$. That is,

$$\frac{1}{M_s} B_s \notin D \Rightarrow B_{s'} \notin D \text{ for some } 0 < s' \leq s. \tag{3.3}$$

Recalling that $\widetilde{B}_s = \frac{1}{M_{\alpha_s}} B_{\alpha_s}$ and that $\alpha_s \le s$ for all $s > 0$, we can rewrite (3.3) as follows

$$\widetilde{B}_s \notin D \Rightarrow B_{s'} \notin D \text{ for some } 0 < s' \le \alpha_s \le s. \tag{3.4}$$

This in turn is equivalent to

$$\tau_{\gamma_2} \le \alpha_{\widetilde{\tau}_{\gamma_2}} \le \widetilde{\tau}_{\gamma_2}, \tag{3.5}$$

where τ_{γ_2} and $\widetilde{\tau}_{\gamma_2}$ denote the killing times of B_t, respectively $\widetilde{B}_t$, on the curve γ_2. From this, it follows that we have

$$P^{r_1 e^{i\theta}} \{\tau_{\gamma_2} > t\} \le P^{r_2 e^{i\theta}} \{\widetilde{\tau}_{\gamma_2} > t\}. \tag{3.6}$$

Thus the function $u(z) = P^z \{\tau_D > t\}$ is nondecreasing on the part of the radii $r_\theta = \{re^{i\theta}, 0 < r < 1\}$ which is contained in the domain D. As before, this together with the real analyticity of the function u shows that it is in fact strictly increasing. This completes the proof of the theorem. □

The Corollary 1.2 follows from Theorem 1.1 exactly as in [5]. Briefly, by Proposition (3.5) of [5],

$$P^z\{\tau_D > t\} = e^{-\mu_1 t}\psi_1(z) \int_D \psi_1(w)dw + \int_D R_t(z,w)dw, \tag{3.7}$$

where

$$e^{\mu t} R_t(z,w) \to 0,$$

as $t \to \infty$, uniformly in $z,\ w \in D$. From this it follows that if γ_2 is an arc of a circle, the function ψ is nondecreasing on the hyperbolic radii γ_θ and that if γ_1 is and arc of a circle the function ψ is nondecreasing along the part of the radii r_θ which are in the domain. The strict increasing follows from the real analyticity. This proves Corollary 1.2. □

In our application of Theorem 2.3, the strict increasing was not really used as this was derived from the fact that quantities involved are solutions of "nice" partial differential equations and hence are real analytic. It may be that the strict increasing of the quantity

$$P^{r\zeta} \left\{ \int_0^{\tau_{U_d^+}} V(B_s)ds > t \right\}$$

can also be proved by relating it to an appropriate PDE.

We end with some other remarks related to Theorem 2.3. Consider the Schrödinger operator $\frac{1}{2}\Delta u - Vu$ in U_d^+ with Dirichlet boundary conditions

on the part of ∂U_d^+ lying in the hyperplane $H = \{(\zeta_1, ...\zeta_d) \in \mathbb{R}^d : \zeta_n = 0\}$, and Neumann boundary conditions on the "top" portion of the sphere. If we let $P_t^V(\xi, \zeta)$, $\xi, \zeta \in U_d^+$ be the heat kernel for this problem, then

$$u(\xi) = E^{\xi}\left\{e^{-\int_0^t V(B_s)ds}\,; \tau_{U_d^+} > t\right\} = \int_{U_d^+} P_t^V(\xi, \zeta) d\zeta.$$

It would be interesting to investigate (under suitable assumptions on V) the monotonicity properties for the function $u(\xi)$. This will lead to "hot–spots" results for the Schrödinger operator defined above. We also refer the reader to [6] where a related problem is studied for the Dirichlet Schrödinger semigroup (in that case one has that near the boundary, and for large values of $t > 0$, the function corresponding function $u(\xi)$ decreases).

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